

# Super-Activation of Quantum Steering

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We consider Einstein-Podolsky-Rosen steering in the regime where the parties can perform collective measurements on many copies of a given shared entangled state. We derive a simple and efficient condition for guaranteeing that an entangled state is  $k$ -copy steerable. In particular we show that any two-qubit and qubit-qutrit entangled states are  $k$ -copy steerable. This allows us to discuss the effect of super-activation of steering, whereby an entangled state that is unsteerable (*i.e.*, admits a local hidden state model) in the one-copy regime becomes  $k$ -copy steerable. We provide examples with few copies and low dimensions. Our results give evidence that entanglement and steering could become equivalent in the multi-copy regime.

## I. INTRODUCTION

The notion of nonlocality appears in quantum mechanics in several forms. Bell nonlocality [1, 2] is the strongest form of inseparability and is relevant to device-independent quantum information protocols [3]. Two distant observers can observe quantum nonlocality by performing local measurements on a shared entangled state, in the sense that the resulting statistics cannot arise from a local hidden variable (LHV) model. Another concept is Einstein-Podolsky-Rosen steering [4, 5], or simply quantum steering, a form of nonlocality that is intermediate between entanglement and Bell nonlocality. Here the measurement statistics are tested with respect to a local hidden state (LHS) model, which represents a particular class of LHV models where the hidden variable can be considered as a quantum state. Steering also finds applications in quantum information [6].

Interestingly these different forms of non separability are strictly different at the level of quantum states [7, 8]. Specifically, there exist entangled states which cannot lead to steering, *i.e.*, admitting a LHS model [5]. Furthermore, there exist entangled states that can lead to steering but cannot give Bell nonlocality, *i.e.*, admit a LHV model [5]. In fact, there is a strict hierarchy between Bell Nonlocality, steering, and entanglement [9], even when considering the most general measurements, *i.e.*, positive-operator-valued-measures (POVMs).

These results were established in the so-called standard Bell/steering scenario, where the parties perform local measurements on a single copy of the entangled state  $\rho$  in each round of the test. In order to obtain the statistics for testing a Bell or steering inequality, many rounds are required. However, one may consider more general Bell/steering tests. In particular, we will consider here the situation where the parties use  $k$  copies of the entangled state  $\rho$ , which can be described as a

global bipartite entangled state  $\rho^{\otimes k}$ , and perform local collective measurements on it. This will be referred to as the  $k$ -copy Bell/steering scenario.

Importantly, the parties can now perform locally some joint measurements (*e.g.*, featuring entangled POVM elements) on  $k$  local systems. This leads to novel possibilities. Namely, it can be the case that an entangled state leads to Bell nonlocality or steering in the  $k$ -copy scenario (*i.e.*,  $\rho^{\otimes k}$  is nonlocal/steerable), despite the fact

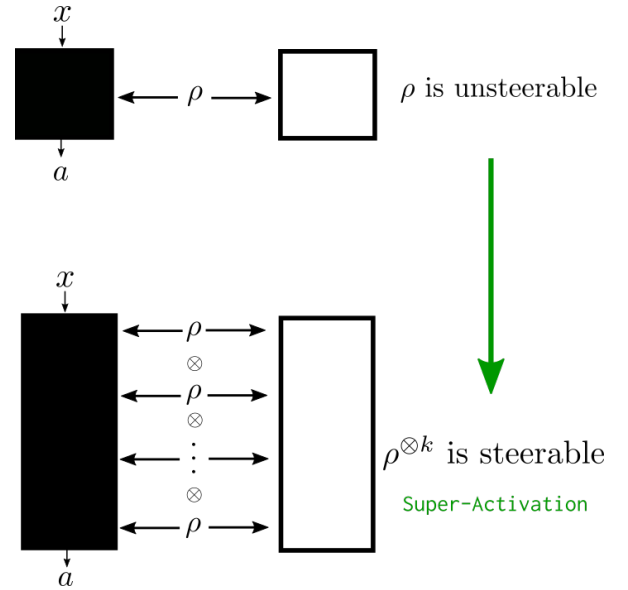


Figure 1. Two parties can obtain quantum steering by performing local measurements on  $k$  copies of an entangled state that cannot lead to steering for a single copy. This phenomenon is referred to as the super-activation of steering. We consider the situation where Alice steers Bob. That is, Alice is the untrusted part (represented by a black box) while Bob is the trusted part characterizing his assemblage via quantum tomography (represented by a white box).

the initial entangled state  $\rho$  admits a LHV/LHS model. This phenomenon is called super-activation of nonlocality/steering. This is possible because the statistics of the joint measurement on  $k$  copies of  $\rho$  are not covered by the LHV/LHS model for  $\rho$  which considers only measurements on a single copy.

In a seminal paper, Peres considered this question, however in a slightly more general scenario, where the parties can perform an initial pre-processing on many copies of the state [10] (see also [11]). In this case, any entangled state that is distillable becomes nonlocal. The scenario without pre-processing, which we focus on here, was only considered later. First, Liang and Doherty [12] showed that the maximal violation of the CHSH inequality [13] of a particular state  $\rho$  can be increased if the parties perform joint measurements on multiple copies. A next step was made by Navascués and Vértesi [14] who presented entangled states such that  $\rho$  does not violate the CHSH inequality but  $\rho \otimes \rho$  does. These results demonstrate the activation of quantum nonlocality, which was also discussed in the multipartite case [15, 16]. Other works showed the activation of general nonlocal non-signaling correlations via local wirings, a process termed nonlocality distillation [17–19].

An even stronger form of activation was discovered by Palazuelos, who showed that quantum nonlocality can be super-activated [20]. Specifically, he proved that certain entangled states  $\rho$  admitting a LHV model for general POVMs can be super-activated, in the sense that  $\rho^{\otimes k}$  violates a Bell inequality (for some finite  $k$ ). To derive this result, he took advantage of known Bell inequalities with unbounded quantum violation [21–23]. Next, Cavalcanti and colleagues [24] presented a general criterion for  $k$ -copy nonlocality. Specifically any entangled state  $\rho$  of dimension  $d \times d$  that is useful for quantum teleportation (*i.e.*, with maximal entanglement fraction greater than  $1/d$  [25]) is  $k$ -copy nonlocal. This construction provides large classes of entangled state for which nonlocality can be super-activated. In particular this is the case for any isotropic entangled state (of any dimension) which admits a LHV model.

In this paper we discuss the  $k$ -copy scenario for quantum steering. Our main result is a simple and general criterion for  $k$ -copy steerability of a quantum state  $\rho$ . Our method exploits the reduction criterion [26], used in the context of entanglement distillation, and can thus be computed efficiently, by finding the minimal eigenvalue of an operator. This result implies that any two-qubit and any qubit-qutrit entangled state is  $k$ -copy steerable. Therefore, given previous results constructing LHS models for such entangled states [5, 27–32], we obtain many examples of super-activation of steering. Our results also allows us to put bounds on the minimal number of copies required for super-activation of steering and nonlocality. We discuss the connection of

our results to entanglement distillation and conclude with some open questions. More generally, our results suggest that  $k$ -copy steerability may be generic for all entangled states (we prove it for  $2 \times 2$  and  $2 \times 3$  entangled states). This would demonstrate the equivalence of entanglement and steering in the multi copy scenario.

## II. BELL NONLOCALITY AND QUANTUM STEERING

Consider two distant observers, Alice and Bob, that can perform local measurements on a shared entangled quantum state  $\rho$ . As standard, quantum measurements are described by the set of POVMs  $\{A_{a|x}\}$  and  $\{B_{b|y}\}$ , positive operators that sum to identity, where  $x$  and  $y$  represent the choice of the measurements and  $a$  and  $b$  their corresponding outcomes. The resulting statistics are given by

$$p(ab|xy) = \text{tr}(\rho A_{a|x} \otimes B_{b|y}). \quad (1)$$

The above distribution is Bell local when it admits a decomposition of the form

$$p(ab|xy) = \int \pi(\lambda) p_A(a|x, \lambda) p_B(b|y, \lambda) d\lambda, \quad (2)$$

where  $\lambda$  is some classical random variable, distributed according to density  $\pi(\lambda)$ , and  $p_A(a|x, \lambda)$  and  $p_B(b|y, \lambda)$  represent the local response functions. A quantum state  $\rho$  is said to be Bell local, or equivalently to admit a LHV model, if its statistics admit a decomposition of the form of Eq. 2 for *all* possible measurements. If such a decomposition does not exist, the state is Bell nonlocal and violates at least one Bell inequality for some choice of local measurements  $\{A_{a|x}\}$  and  $\{B_{b|y}\}$  [1, 2].

A different notion of nonlocality is that of steering. In a steering test, Bob, who does not trust Alice, wants to verify that a state  $\rho$  is entangled. To this end, he asks Alice to perform measurement  $x$  on her subsystem and announce its result  $a$ . By doing so, she remotely steers the state of Bob's system to

$$\sigma_{a|x} = \text{tr}_A(A_{a|x} \otimes I \rho), \quad (3)$$

where  $\text{tr}_A$  denotes the partial trace over Alice's system. Bob's task is now to ensure that the set of unnormalized conditional states  $\{\sigma_{a|x}\}$ , a so-called *assemblage*, does not admit a decomposition of the form

$$\sigma_{a|x} = \int \pi(\lambda) p_A(a|x, \lambda) \rho_\lambda d\lambda, \quad (4)$$

where  $\lambda$  is a classical random variable distributed according to  $\pi(\lambda)$ ,  $p_A(a|x, \lambda)$  represents Alice's response function, and  $\rho_\lambda$  the (hidden) quantum state. If the assemblage observed by Bob (that can be obtained via quantum tomography) does admit a decomposition (4),

Bob concludes that Alice could have cheated by using the following strategy: Alice sends the single party, hence not entangled, quantum state  $\rho_\lambda$  to Bob, and announces the measurement outcome  $a$  according to  $p_A(a|x, \lambda)$ . Note that  $\lambda$  can be understood as Alice's choice of strategy. If an entangled state  $\rho$  admits a decomposition of the form of Eq. (4) for quantum measurements, the state is termed as unsteerable (or equivalently, it admits a LHS model). However, if the assemblage  $\{\sigma_{a|x}\}$  does not admit a decomposition of the form (4) for some set of measurements  $A_{a|x}$ , the state is said to be steerable and it violates a steering inequality [33].

We note that a LHS model is a particular case of a LHV model in which Bob's response functions respects the trace rule imposed by quantum mechanics (see [34]). Hence, states admitting a LHS model always admit a LHV one, whereas the converse does not hold in general [9]. We also point out that the asymmetry of steering allows the existence of entangled states which are only one-way steerable [9, 35].

In the next sections, we will make use of a quantification of bipartite quantum entanglement. The *maximal entanglement fraction* [25]  $F$  of a state  $\rho$  is defined as the maximal overlap of  $\rho$  with any maximally entangled state. More precisely,

$$F(\rho) := \max_{U_A, U_B} \text{tr}(\rho U_A \otimes U_B |\phi_d^+\rangle\langle\phi_d^+| U_A^\dagger \otimes U_B^\dagger), \quad (5)$$

where  $U_A$  and  $U_B$  are local unitary operators and  $|\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$  the  $d$ -dimensional maximally entangled state.

The concept of maximal entanglement fraction is intimately related to *isotropic states* [26], quantum states with a high level of symmetry. A  $d$ -dimensional isotropic state with entanglement fraction  $F$  is defined as

$$ISO_d(F) = F |\phi_d^+\rangle\langle\phi_d^+| + (1 - F) \frac{I_{d \times d} - |\phi_d^+\rangle\langle\phi_d^+|}{d^2 - 1}. \quad (6)$$

It can be shown that any quantum state  $\rho$  acting in  $\mathbb{C}^d \otimes \mathbb{C}^d$  with maximal entanglement fraction  $F$  can be transformed into a  $d$ -dimensional isotropic state with entanglement fraction  $F$  by local unitary operations and shared randomness via an *isotropic twirling* operation [26]. Since local unitary transformation and shared randomness cannot create nonlocality (steering nor Bell), it follows that if a state  $\rho$  can be transformed into a steerable (Bell nonlocal) isotropic state,  $\rho$  is guaranteed to be steerable (Bell nonlocal).

### III. STEERABILITY OF MULTIPLE COPIES

Our goal now is to discuss steering in the  $k$ -copy scenario. Specifically, given an initial entangled state  $\rho$ ,

we will consider the assemblage

$$\sigma_{a|x} = \text{tr}_A(A_{a|x} \otimes I \rho^{\otimes k}), \quad (7)$$

where  $A_{a|x}$  now represents a local measurement of Alice performed on all  $k$  subsystems she holds. In particular, these can be any possible joint measurement. In case one can find a set of measurements  $A_{a|x}$  such that the resulting assemblage is steerable, we say that the state  $\rho$  is  $k$ -copy steerable (from Alice to Bob).

We now present our main result, which is a simple and general criterion for ensuring that an entangled state  $\rho$  is  $k$ -copy steerable. In particular, this result then implies that  $k$ -copy steerability is generic for two-qubit and qubit-qutrit entangled states.

**Theorem 1.** *All quantum states  $\rho$  acting on  $\mathbb{C}^d \otimes \mathbb{C}^d$  for which the operator*

$$I_d \otimes \rho_B - \rho \quad (8)$$

*is not positive definite (i.e.,  $\rho$  violates the reduction criterion) are  $k$ -copy steerable from Alice to Bob for some number of copies  $k$ .*

*Moreover, let  $F$  be the maximal entanglement fraction of  $\rho$ . Then,  $\rho$  is  $k$ -copy steerable whenever*

$$F^k > \left[ (1 + d^k) \left( \sum_{i=1}^{d^k} \frac{1}{i} - 1 \right) - d^k \right] \frac{1}{d^{2k}}. \quad (9)$$

We note that the *reduction criterion* [26, 36] gives a sufficient condition for entanglement distillability. The connection to this topic will be discussed in Section V.

*Proof.* The first step of the proof makes use of properties of assemblages admitting a LHS model. Specifically, we recall that local unitary operations, shared randomness, and local filtering operations on Bob's side preserve unsteerability [9, 29, 37]. More precisely, one has the following statement.

**Lemma 1.** *Let  $\rho$  be an entangled state, unsteerable from Alice to Bob. For any local operation represented by a positive map  $\Lambda$  acting on Bob's system, the final state*

$$\rho_F = \frac{(I \otimes \Lambda)(\rho)}{\text{tr}[(I \otimes \Lambda)(\rho)]} \quad (10)$$

*is unsteerable from Alice to Bob.*

We refer the reader to Refs [9, 29] for the proof.

The second step consists in using the reduction criterion. Note that any entangled state  $\rho$  for which the operator  $I_d \otimes \rho_B - \rho$  is not positive definite, can be transformed into a  $d$ -dimensional isotropic state  $ISO_d(F)$  with  $F > 1/d$  via an isotropic twirling operation, which consists of local unitaries and shared randomness [26]. Therefore, for any state  $\rho$  violating the reduction criterion, we can construct a local filter on Bob's side that

will map  $\rho$  to an isotropic state  $ISO_d(F)$  with  $F > 1/d$  (after isotropic twirling); see Appendix for details on the local filter. Since the latter is  $k$ -copy Bell nonlocal, following the result of Ref. [24], it follows that it is also  $k$ -copy steerable; this is because Bell nonlocality always implies steering. Therefore, it follows from Lemma 1 that  $\rho$  is  $k$ -copy steerable.

We also give a more refined proof that requires in general less copies of the initial state  $\rho$ . We observe that  $k$ -copies of a  $d$ -dimensional isotropic state have the form

$$[ISO_d(F)]^{\otimes k} = F^k |\phi_d^+\rangle\langle\phi_d^+|^{\otimes k} + (1 - F^k)\rho_R, \quad (11)$$

$$= F^k |\phi_{d^k}^+\rangle\langle\phi_{d^k}^+| + (1 - F^k)\rho_R, \quad (12)$$

where the state  $\rho_R$  is orthogonal to the maximally entangled one, i.e.,  $\text{tr}(|\phi_{d^k}^+\rangle\langle\phi_{d^k}^+|\rho_R) = 0$ . Hence, we can transform  $k$  copies of a  $d$ -dimensional isotropic state with entanglement fraction  $F$  into a single copy of a  $d^k$ -dimensional isotropic state with entanglement fraction  $F^k$  via an isotropic twirling operation, that is,

$$[ISO_d(F)]^{\otimes k} \mapsto ISO_{d^k}(F^k). \quad (13)$$

Next, we exploit a result given in Ref. [5], where the authors presented a necessary and sufficient condition for steerability of a  $d$ -dimensional isotropic state, considering all projective measurements. This criterion can be expressed in terms of the maximal entanglement fraction. Specifically, the state is steerable if and only if

$$F > F_{proj} = \frac{\left[(1+d) \sum_{n=1}^d \frac{1}{n}\right] - d}{d^2}. \quad (14)$$

Applied to our case, we obtain a sufficient condition for  $k$ -copy steerability of the initial state. More precisely, any state  $\rho$  with maximal entanglement fraction  $F$  is  $k$ -copy steerable whenever

$$F^k > \frac{(1+d^k)(\sum_{i=1}^{d^k} \frac{1}{i} - 1) - d^k}{d^{2k}}. \quad (15)$$

□

The above result leads to a general result for all qubit-qubit and qubit-qutrit entangled states, as all these states violate the reduction criterion [26, 38]. Hence we have that

**Corollary 1.** *All entangled  $2 \times 2$  entangled state are  $k$ -copy (two-way) steerable, and all  $2 \times 3$  entangled states are  $k$ -copy steerable (from the qubit to the qutrit part).*

#### IV. SUPER-ACTIVATION OF QUANTUM STEERING

In the previous sections we have discussed how to reveal steering of a quantum state  $\rho$  by performing measurements on many copies of it. We are thus now

all set up to address the question of super-activation. That is, we look for states which admit a LHS model in the single copy regime but violate a steering inequality when  $k$  copies are considered. More precisely we focus on the question: “Can we find a state  $\rho$  which admits a LHS model, such that  $\rho^{\otimes k}$  becomes steerable?”

Already the results of the previous section can directly be employed to reveal super-activation of quantum steering. First of all, the fact that every entangled  $2 \times 2$  state is steerable for some number of copies  $k$ , together with the fact that there are numerous  $2 \times 2$  entangled state admitting a LHS model [5, 27–32] is sufficient to show super-activation. The same is true of course for the case of  $2 \times 3$  entangled states admitting a LHS model (from Alice to Bob). More generally, any entangled state admitting a LHS and violating the reduction criterion will feature super-activation of steering.

Moreover, we note that the mere existence of the super-activation of steering already follows from the works of Refs [20] and [24]. This is due to the fact that (i)  $k$ -copy nonlocality implies  $k$ -copy steerability (two-way), and (ii) some of the entangled states discussed in Refs [20, 24] admit a LHS model. Nevertheless, the present results are stronger, in the sense that our criterion can detect much larger classes of entangled states compared to Refs [20, 24]. For instance, our criterion detects all two-qubit entangled states, which is not the case for Refs [20, 24]. Moreover, our results also provide much stronger bounds on the minimal number of copies  $k$  required for super-activation as we will discuss now.

##### A. Few copies and low dimensions

While we have discussed above several examples of the super-activation of steering, we ask now a more sophisticated questions, looking for minimal examples of super-activation. Specifically, we ask the question: “Let  $\rho$  be a state that can be super-activated. What is the minimum number of copies  $k$  for the final state  $\rho^{\otimes k}$  to be steerable?” Below we give example of super-activation of steering with few copies and construct an example where  $k = 2$  copies are enough. Moreover, we also give examples for states of low dimension.

We start with the  $d$ -dimensional isotropic state with entanglement fraction  $F$

$$ISO_d = F |\phi_d^+\rangle\langle\phi_d^+| + (1 - F) \frac{I_{d \times d} - |\phi_d^+\rangle\langle\phi_d^+|}{d^2 - 1} \quad (16)$$

that is known to have an LHS model for projective measurements if and only if [5]

$$F \leq \frac{\left[(1+d) \sum_{n=1}^d \frac{1}{n}\right] - d}{d^2}, \quad (17)$$



and to have a LHS model for POVMs if [39]

$$F \leq \frac{1 + \left(\frac{d+1}{d}\right)^d (3d-1)}{d^2}. \quad (18)$$

Using theorem 1 one can explicitly find sufficient requirements on the dimension  $d$  and number of copies  $k$  for obtaining super-activation of steering. We start with the case  $d = 2$ . From the above inequalities (17) and (18) we see that if Alice and Bob share  $k$  copies of a two-qubit isotropic state, they can super-activate steering for projective measurements with  $k = 7$  copies, and super-activate steering for POVMs with  $k = 24$  copies. Next we discuss the case of only  $k = 2$  copies. Here, considering projective measurements, we get that steering is super-activated for isotropic states of dimension  $d \geq 6$ . However, when considering POVMs, the combination of theorem 1 and condition (18) does not allow us to prove super-activation directly.

Moreover, we remark that the criterion for  $k$ -copies steerability presented in equation (9) is always more “economical” in number of copies than the criterion presented in Ref. [24]. This is due to the fact that both proof methods rely on the nonlocality of an isotropic state, but while Ref. [24] focuses on a specific Bell inequality, namely the Khot-Vishnoi game [22, 23], our method considers *all* steering inequalities for projective measurements via the necessary and sufficient criterion presented in [5].

Before finishing this section we point out a simple method to show super-activation of steering (or Bell nonlocality) for the scenario where the parties share only two copies of an initial state  $\rho$ . That is, there exists  $\rho$  which has a LHS (LHV) model for general POVMs and  $\rho \otimes \rho$  violates a steering (Bell) inequality. While this method is implicitly suggested in Palazuelos’ paper [20], we present it here in full detail for completeness

1. Choose  $\rho$  acting on  $\mathbb{C}^d \otimes \mathbb{C}^d$  that has a LHS (LHV) model for POVMs, and that is  $k$ -copy steerable (Bell nonlocal).
2. There exists a critical number of copies  $k_C > 1$  such that  $\rho^{\otimes k_C}$  is unsteerable (Bell local), but  $\rho^{\otimes k_C+1}$  is steerable (Bell nonlocal)
3. By construction, the new state  $\rho' = \rho^{\otimes k_C}$  is unsteerable (Bell local) but two copies of it violate a steering (Bell) inequality.

Following the above protocol, one can transform any example of  $k$ -copy super-activation of steering (Bell nonlocality) into another one that requires only 2 copies. The “price” to pay for this construction is that the dimension of the quantum state  $\rho' = \rho^{\otimes k_C}$  that exhibits super-activation with two copies becomes larger than the initial one.

## V. CONNECTION TO ENTANGLEMENT DISTILLATION

Two parties sharing many copies of an entangled state  $\rho$  and having access to local operations and classical communication (LOCC) can in some cases obtain (or more precisely, arbitrarily approximate) a maximally entangled state via an *entanglement distillation protocol* [40, 41].

Clearly, entanglement distillation shares similarities with the scenario of  $k$ -copy steering. The notable difference is however that LOCC is not considered in the latter. Thus, the fact that a given state is distillable does not necessarily imply that it will lead to  $k$ -copy steering, since LOCC is not considered in  $k$ -copy steering.

More generally, the link between entanglement distillation and nonlocality in asymptotic scenarios has been discussed. First, it was conjectured by Peres [10] that all entangled but undistillable states, *i.e.*, bound entangled states, admit a LHV model Bell even in the  $k$ -copy regime with local pre-processing. This conjecture was further strengthened to the case of an LHS model [42]. Nevertheless, these conjectures are known to be false since there exist bound entangled states that can lead to steering [43] and Bell nonlocality [44]. In fact, these examples consider the standard single copy scenario. This shows that entanglement distillation and nonlocality are different in general.

### A. Connection to One-Way Distillation

It is worth pointing out a connection between *one-way entanglement distillation* and  $k$ -copy steering. A state  $\rho$  is said to be one-way distillable when it can be distilled via only one-way communication, say from Alice to Bob.

The connection goes as follows. The existence for a one-way distillation protocol for some state  $\rho$  always implies that a maximally entangled state can also be obtained via a local filtering operation on one side only [41]. Since this class of operations cannot generate steering (see Lemma 1), it follows that *all one-way distillable states are  $k$ -copy steerable*.

This allows us to detect  $k$ -copy steerable states using known criterion of one-way distillability, such as the one-way hashing protocol [45]. The latter says that any state  $\rho$  that satisfies  $S(\rho_B) > S(\rho)$ , with  $S(\rho)$  being the Von-Neumann entropy of  $\rho$ , are one-way distillable. Hence, any state  $\rho$  detected by the one-way hashing protocol is  $k$ -copy steerable from Alice to Bob.

## VI. DISCUSSION

In this paper we have explored steering in a scenario where the parties can perform joint measurements on many copies of a given entangled state. We gave a general and simple criterion to detect entangled states that exhibit  $k$ -copy steering. Notably, this criterion detects all entangled states in dimension  $2 \times 2$  and  $2 \times 3$ . Moreover, we discussed several examples of super-activation of steering, some of which involving few copies and low dimensions. Finally, we connected this problem to entanglement distillation.

The natural question now would be: are all entangled states  $k$ -copy steerable? Although we answered this question positively for the particular case of qubit-qubit and qubit-qudit systems, there may exist some entangled qudit-qudit states that admits a LHS or LHV model even when an arbitrary number of copies is considered. This is similar in spirit to the still unresolved question of  $k$ -copy non-locality. While previous conjectures ruled out the possibility based on a connection to distillability [10, 42], the fact that undistillable states can exhibit steering [43] and even non-locality [44] implies that the question is still essentially open.

Another question is whether super-activation of steering necessarily implies the super-activation of nonlocality. For instance, could one find an entangled two-qubit state admitting an LHV model for any number of copies? The connection to entanglement distillability also deserves interest. Are all distillable states  $k$ -copy steerable (nonlocal)?

## VII. ACKNOWLEDGEMENTS.

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## NOTE ADDED

While completing this manuscript, we became aware of the work of [46] who independently derived a suffi-

cient condition for steerability in terms of the fully entangled fraction and demonstrated the super-activation of steering for the isotropic states.

## Appendix A: More Details On The proof of Theorem 1

We now reproduce a lemma first presented in Ref. [26] that we used for proving our main result.

**Lemma 2.** *Let  $\rho$  be a bipartite state acting on  $\mathbb{C}^d \otimes \mathbb{C}^d$  which  $I \otimes \rho_B - \rho$  is not positive definite. There exists a local filter operation  $F_B$  such that the filtered state*

$$\rho' = \frac{I \otimes F_B \rho I \otimes F_B^\dagger}{\text{tr}(I \otimes F_B \rho I \otimes F_B^\dagger)} \quad (\text{A1})$$

satisfies  $\text{tr}(\rho' |\phi_d^+\rangle\langle\phi_d^+|) > 1/d$ .

*Proof.* If the operator  $I \otimes \rho_B - \rho$  is not positive definite, there exists a pure quantum state  $|\psi\rangle = \sum_{ij} \alpha_{ij} |ij\rangle$  such that

$$\text{tr}((I \otimes \rho_B - \rho) |\psi\rangle\langle\psi|) < 0 \quad (\text{A2})$$

or equivalently,  $\text{tr}(\rho |\psi\rangle\langle\psi|) > \text{tr}(\rho_B \psi_B)$ , where  $\psi_B = \text{tr}_A(|\psi\rangle\langle\psi|)$ . We now define the local filter  $F_B := \sqrt{d} \sum_{ij} \bar{\alpha}_{ij} |i\rangle\langle j|$ , with  $\bar{\alpha}_{ij}$  being the complex conjugate of  $\alpha_{ij}$ . With that, we see that  $|\psi\rangle = I \otimes F_B^\dagger |\phi_d^+\rangle$ , and construct the filtered state is

$$\rho' := \frac{I \otimes F_B \rho I \otimes F_B^\dagger}{\text{tr}(I \otimes F_B \rho I \otimes F_B^\dagger)}. \quad (\text{A3})$$

Now, a straightforward calculation shows that

$$\begin{aligned} F_B^\dagger F_B &= d \rho_B, \\ \text{tr}(I \otimes F_B \rho I \otimes F_B^\dagger) &= d \text{tr}(\rho_B \psi_B), \\ \text{tr}(I \otimes F_B \rho I \otimes F_B^\dagger |\phi_d^+\rangle\langle\phi_d^+|) &= \text{tr}(|\psi\rangle\langle\psi| \rho), \end{aligned} \quad (\text{A4})$$

and since we have  $\text{tr}(\rho |\psi\rangle\langle\psi|) > \text{tr}(\rho_B \psi_B)$  by hypothesis, it follows that

$$\text{tr}(\rho' |\phi_d^+\rangle\langle\phi_d^+|) > 1/d. \quad (\text{A5})$$

□

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